Int J Theor Phys (2008) 47: 1459–1470
DOI 10.1007/s10773-007-9588-7

Stochastically and Intrinsically Extended Non-relativistic Quantum Particles

A. Smida · M. Hachemane · A.-H. Hamici · Y. Oualili

Received: 13 May 2007 / Accepted: 5 October 2007 / Published online: 23 October 2007

Abstract Stochastically and intrinsically extended non relativistic quantum particles are described by combining the ideas of a stochastic quantum theory and a quantum functional theory. The former relates the extension to imperfect real measurements while the latter considers it as intrinsic. Physical states, Positive-Operator-Valued measures connected to measurement, and propagators are given and discussed. The stochastic theory is sufficient when the bilocal field describing the particle has a product form.

Keywords Positive-Operator-Valued measures · Extended particles · Stochastic theory · Quantum mechanics

1 Introduction

Conventional quantum mechanics is based on a pointlike conception of elementary particles. This conception extrapolates to the relativistic regime and to conventional quantum field theory although it has been contested from the early days of quantum mechanics. De Broglie tried to conceive the pointlike image as a singularity in a physical wave u which represents an extended body [5]. Thereafter, Destouches proposed a generalized version of this idea in his functional theory [6, 7]. Its main feature is that the analysis of the concept of physical system with respect to the remaining part of the Universe leads to the influence of the latter on the intrinsic characteristics of the former. As a consequence, an elementary particle may be represented by a function u describing these characteristics and, as such, it must be conceived as a nonrigid extended body. This replacement of the pointlike conception $\mathbf{x} \in \mathbf{R}^3$ by a functional conception u entails a replacement of the conventional quantum mechanical wave function $\hat{\psi}_t(\mathbf{x}) = \hat{\psi}(t, \mathbf{x})$ by a quantum functional wave $X_t[u] = X(t, u)$. The state X may not belong to a Hilbert space (but to a space which contains one) and the physical wave u may be handled by associating it to a (realistic) model. Of course, abiding by a realistic standpoint is comforting but we prefer taking a detached point of view by replacing

A. Smida · M. Hachemane (✉) · A.-H. Hamici · Y. Oualili
Faculté de Physique, USTHB, B.P. 32 El-Alia Bab-Ezzouar, Alger 16111, Algeria
e-mail: hachemane@wissal.dz

the term (realistic) by the more general term (physical). In other words, what is important for us is the banishment of the pointlike conception of elementary particles. A geometro-differential model [10, 17] has been constructed on this basis, where the extended particles are composed of two quantum local modes. The external mode evolves in the external physical space-time and the internal mode is localized in an internal space-time, the points of which are arguments of the physical wave u. Let us call these particles *intrinsically* or *functionally extended* as opposed to those which are *stochastically extended* and introduced in a recent (geometro)-stochastic quantum theory [14–16]. The stochastic theory [14] stemmed from group theoretic considerations [2, 3] linked with measurement theoretic ones [13], and is based on an operational principle asserting that a physical theory should take into account the experimental circumstances surrounding the observation process. The main property of concern is the imperfect nature of the measuring apparatuses. In this context, the position $\mathbf{q}$ and momentum $\mathbf{p}$ of a system particle are determined with confidence functions $\hat{\chi}_{\mathbf{q}}$ and $\tilde{\chi}_{\mathbf{p}}$ reflecting the imperfectness of real measuring devices. Respectively, these confidence functions are squares of two functions $\hat{\xi}$ (in the configuration representation) and $\tilde{\xi}$ (in the momentum representation) that specify irreducible phase space representations of the Galilean group [14]. Moreover, translating their phase space representative ξ with all amounts $\mathbf{q}$ and boosting it to all velocities $\mathbf{v} = \mathbf{p}/m$ leads to an overcomplete family $\{\xi_{\mathbf{q},\mathbf{p}}; \mathbf{q} \in \mathbf{R}^3, \mathbf{p} \in \mathbf{R}^3\}$ of generalized coherent states in the phase space representation Hilbert space. Correspondingly, the element ξ is interpreted as a proper state vector of a test particle playing the role of a microdetector which is stochastically at rest at the origin of an inertial frame ([13, 14]; see relations (10) and (12) below). The element $\xi_{\mathbf{q},\mathbf{p}}$ corresponds to a state of the same particle with stochastic position $(\mathbf{q}, \hat{\chi}_{\mathbf{q}})$ and momentum $(\mathbf{p}, \tilde{\chi}_{\mathbf{p}})$. The whole family constitutes a quantum frame lifting the status of space-time to a quantum level in a fiber bundle geometric structure capable of bearing a consistent formulation of quantum gravity [15, 16].

Our main problem is the reconsideration of the functional and stochastic theories from their fundamental principles in the nonrelativistic regime. We think that both are so natural that they must be taken into account on an equal footing. In fact, the interpretation of the marginal probabilities, of measuring only the position or momentum of the particle, is fundamental in the stochastic theory. According to that interpretation, a measurement is performed with a real (imperfect) apparatus on a system particle that retains an intrinsic pointlike nature since its stochastic extension is a mere reflection of that imperfectness. In contradistinction, the system particle is intrinsically extended in the functional theory but the functionals give previsions for the measurement of sharp (accurate) values of observables with a perfect apparatus. Since the adoption of an intrinsic pointlike image for the system particle and a perfect nature of the apparatus are equal idealizations, we shall try to construct a model that combines the functional and the stochastic theories so that the particles are extended in the intrinsic and stochastic senses. The discussion of specific advantages of such a construction will be shifted to the end of the conclusion since it is connected with future developments in the relativistic case. At the present stage, we prefer checking the theory in the non relativistic regime (where the interpretations are clear-cut) and not to overshadow its generality with specific applications since they may be very diverse.

The state of the system particle is represented by the functional $X[\hat{u}] = \hat{\Psi}[\hat{u}]$ identified with a bilocal field $\hat{\Psi}(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in \mathbf{R}^3$ is the external space variable and $\mathbf{y} \in \mathbf{R}^3$ is the internal one. We shall proceed gradually in considering the test particle (microdetector) and the system particle. One first natural assumption is that the external part of the intrinsically extended system particle is observed with the stochastic test particles, while the internal mode retains its pointlike character. Hence, the stochastic proper state vectors $\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}$ display a pointlike character with respect to $\mathbf{y}$. A little more involved assumption is that the test particles have an intrinsic extension in addition to the stochastic one, with functional proper state vector $\hat{\Xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}[\hat{\lambda}] = \hat{\xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}\hat{\lambda}$ being the product of their stochastic proper state $\hat{\xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}$ (or its generalized version) with their physical wave $\hat{\lambda}$. This case has some analogy with the (statistical formulation of) stochastic theory when the test particles have internal structure [14].

In Sect. 1, we introduce our model and reanalyze as briefly as possible the functional and the stochastic theories to make the paper understandable and self-contained. In Sect. 2, we consider the case where the test particle is stochastically extended and the system particle is intrinsically extended. We rewrite expressions for the most fundamental objects. Namely, we define the Hilbert spaces, the proper state vectors and the generic state vectors. The resulting systems of covariance, which are positive operator valued measures giving the probabilities of the position and momentum simultaneous measurement outcomes, are then determined in conjunction with the ensuing marginal probabilities. The formal expression of the free propagator is given also. In Sect. 3, we postulate the system of covariance related to stochastic test particles which have an intrinsic extension. Then, all other expressions are derived for intrinsically extended particles with some remarks for pointlike system particles. In Sect. 4, we conclude our work.

2 Functional and Stochastic Theories

The functional theory replaces the pointlike conception of an elementary particle (a point $\mathbf{x} \in \mathbf{R}^3$) by a functional one whereby this particle is represented by a physical wave u depending on a space-time variable $y = (y^0, \mathbf{y})$. The physical interpretation of $(y^0, \mathbf{y})$ depends on the physical model adopted for the treatment of u. In our geometro-differential model [10], they belong to the internal space-time of the particle while the variables $x = (t, \mathbf{x})$ belong to the physical external space-time being the base manifold of a Hilbert bundle. The elements of the fibers above x are the physical waves u describing the proper characteristics of the particle in an irreducible induced representation of the internal symmetry group [12]. Hence, they were interpreted as representing internal pointlike quantum modes localized in the internal space. According to the functional theory, probabilities are given by the functional wave $X[t, u]$ having a spectral decomposition

$$X = \sum_i c_i X_i \tag{1}$$

with respect to an observable A with a spectrum a_i and probability amplitudes c_i. The elements X_i can be called the *proper functional states* corresponding to a value a_i with probability one. In the strict sense, $X[t, u]$ should be a wave in a space R_u containing the physical waves. Since the guidelines for the definition of such a wave are far from being obvious, we have previously [10, 17] chosen a simple bilocal form

$$X[t, u](\mathbf{x}, y) = \hat{\psi}(x, y) \tag{2}$$

interpreted as the probability amplitude that the external mode be localized at $\mathbf{x}$ at time t and the internal mode be localized at $\mathbf{y}$ at (relative, proper, ...) time y^0. It is clear from the above considerations that albeit the particle is extended, the observed values (such as $\mathbf{x}$) are considered as being sharp. Our aim is to improve this situation with the stochastic theory to the presentation of which we now turn.

Stochastic localization is described by systems of covariance,

$$\mathbf{P}_{\xi}(B) = \int_B |\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p}}|; \quad \mathbf{q},\mathbf{p} \in \Gamma = \mathbf{R}^6 \tag{3}$$

defined as positive operator valued (POV) measures over Borel sets B in nonrelativistic phase space Γ, rather than by systems of imprimitivity which are projector valued measures [12, 14]. The states $|\xi_{\mathbf{q},\mathbf{p}}\rangle$ are obtained from a single state $|\xi\rangle$ by a kinematic operation of translation by an amount $\mathbf{q}$ and a boost to a velocity $\mathbf{v} = \mathbf{p}/m$ where m is the mass of the particle. In the configuration representation, this reads

$$\hat{\xi}_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = (\hat{U}_{\mathbf{q},\mathbf{p}}\hat{\xi})(\mathbf{x}) = \exp\left[\frac{i}{\hbar}\mathbf{p}(\mathbf{x}-\mathbf{q})\right]\hat{\xi}(\mathbf{x}-\mathbf{q}) \tag{4}$$

where $\hat{U}_{\mathbf{q},\mathbf{p}} = \hat{U}\ (b = 0, \mathbf{q}, \mathbf{v} = \mathbf{p}/m, R = I)$ is a representation of a Galilean transformation with no time translation b and no rotation R. The function $\hat{\xi}(\mathbf{x})$ belongs to $L^2(\mathbf{R}^3)$ with norm $\|\hat{\xi}\| = (2\pi\hbar)^{-3/2}$ and is rotationally invariant. It is a configuration representative of the phase space proper state vector ξ which generates a resolution of the identity

$$\int_\Gamma |\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p}}| = \mathbf{1} \tag{5}$$

in the Hilbert space of (phase space representation) state vectors, so that

$$|\psi\rangle = \int_\Gamma d\mathbf{q}d\mathbf{p}\psi(\mathbf{q},\mathbf{p})|\xi_{\mathbf{q},\mathbf{p}}\rangle \tag{6}$$

$$\psi(\mathbf{q},\mathbf{p}) = \langle\xi_{\mathbf{q},\mathbf{p}}|\psi\rangle = \langle\hat{\xi}_{\mathbf{q},\mathbf{p}}|\hat{\psi}\rangle = \langle\tilde{\xi}_{\mathbf{q},\mathbf{p}}|\tilde{\psi}\rangle \tag{7}$$

The above inner products are defined by integrals over phase space, configuration, and momentum spaces respectively. Thus, the irreducible phase space representation space $L^2_\xi(\mathbf{R}^6)$ contains square integrable wave function related to the corresponding configuration and momentum representation wave functions by

$$\psi(\mathbf{q},\mathbf{p}) = \int \hat{\xi}^*_{\mathbf{q},\mathbf{p}}(\mathbf{x})\hat{\psi}(\mathbf{x})d\mathbf{x} = \int \tilde{\xi}^*_{\mathbf{q},\mathbf{p}}(\mathbf{k})\tilde{\psi}(\mathbf{k})d\mathbf{k} \tag{8}$$

When the stochastic particle is in the state $|\psi\rangle$, the probability that a simultaneous measurement of stochastic position and momentum yield values $(\mathbf{q},\mathbf{p})$ within $B \subset \Gamma$, with confidence functions $\hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x})$ and $\tilde{\chi}^\xi_{\mathbf{p}}(\mathbf{k})$ is given by

$$P^\xi_\psi(B) = \langle\psi|\mathbf{P}_\xi(B)\psi\rangle = \int_B d\mathbf{q}d\mathbf{p}|\psi(\mathbf{q},\mathbf{p})|^2 \tag{9}$$

The configuration confidence function $\hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x})$ appears in the marginal probability that the position measurement result $\mathbf{q}$ belong to the set $\Delta_1 \in \mathbf{R}^3$

$$P^\xi_\psi(\Delta_1 \times \mathbf{R}^3) = \int_{\Delta_1} d\mathbf{q}\int_{\mathbf{R}^3} d\mathbf{x}\hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x})|\hat{\psi}(\mathbf{x})|^2 \tag{10}$$

$$\hat{\chi}^\xi_{\mathbf{q}}(\mathbf{x}) = (2\pi\hbar)^3|\hat{\xi}(\mathbf{x}-\mathbf{q})|^2 \tag{11}$$

The marginal probability for the momentum $\mathbf{p}$ to be observed within the set $\Delta_2 \in \mathbf{R}^3$ has an analogous expression

$$P_{\psi}^{\xi}(\mathbf{R}^3 \times \Delta_2) = \int_{\Delta_2} d\mathbf{p} \int_{\mathbf{R}^3} d\mathbf{k} \tilde{\chi}_{\mathbf{p}}^{\xi}(\mathbf{k}) |\tilde{\psi}(\mathbf{k})|^2 \tag{12}$$

$$\tilde{\chi}_{\mathbf{p}}^{\xi}(\mathbf{k}) = (2\pi\hbar)^3 |\tilde{\xi}(\mathbf{k} - \mathbf{p})|^2 \tag{13}$$

The interpretation of $P_{\psi}^{\xi}(\Delta_1 \times \mathbf{R}^3)$ as a probability is based on the interpretation of $\hat{\chi}_{\mathbf{q}}^{\xi}(\mathbf{x})$ as a conditional probability density that the reading of position be $\mathbf{q}$ when the system particle is localized at $\mathbf{x}$, and the interpretation of $|\hat{\psi}(\mathbf{x})|^2$ as a probability density of its localization at $\mathbf{x}$. In this respect, $P_{\psi}^{\xi}(\Delta_1 \times \mathbf{R}^3)$ is the probability that only the reading of the measurement of the position (and not the actual position of the particle) belongs to Δ_1. This accounts for a stochastic localization in configuration space due to the imperfect nature of the measuring device described by the confidence function $\hat{\chi}_{\mathbf{q}}^{\xi}(x)$. This interpretation holds true in the momentum space. Hence, the system particle can be considered as stochastically extended since it can never be sharply localized. The measuring apparatus, which is described by $\hat{\chi}_{\mathbf{q}}^{\xi}(x)$, acquires a quantum character by interpreting the function $\hat{\xi}$ as a proper wave function of a stochastically extended test particle playing the role of a (real or imperfect) microdetector which is at stochastic rest at the origin of a classical system of reference. The state $|\xi_{\mathbf{q},\mathbf{p}}\rangle$ is then interpreted as proper state vector of such a microdetector localized at mean stochastic position $\mathbf{q}$ with mean momentum $\mathbf{p}$. The family $\{|\xi_{\mathbf{q},\mathbf{p}}\rangle; (\mathbf{q}, \mathbf{p}) \in \mathbf{R}^6\}$ corresponds to an array of microdetectors and constitutes a quantum frame [14–16]. The above POV measures are then associated to these microdetectors and describe this type of measurement.

It is clear that the configuration and momentum marginal probabilities are the cornerstone for the consistency of the physical interpretation. However, in these marginal probabilities, the system particle is conceived as essentially pointlike with sharp probability amplitudes $\hat{\psi}(\mathbf{x})$ and $\tilde{\psi}(\mathbf{k})$. The question that we are addressing in this work is how the stochastic formalism should be presented when dealing with intrinsically extended particles. It turns out that the answer depends on how the intrinsic extension is conceived. In some of these conceptions, we have to rewrite the system of covariance in a generalized form introduced in a statistical formulation of the stochastic theory and related to a system of N particles with internal structure [14]. When specialized to the case of one particle, the generalized system of covariance is given by the POV measure

$$\mathbf{P}_{\gamma}(B) = \int_{B\subset\Gamma} \gamma_{\mathbf{q},\mathbf{p}} d\mathbf{q} d\mathbf{p} \tag{14}$$

where $\gamma_{\mathbf{q},\mathbf{p}} = U_{\mathbf{q},\mathbf{p}} \gamma U_{\mathbf{q},\mathbf{p}}^{-1}$ and the operator γ belongs to the trace class and can be written as

$$\gamma = \sum_{i=1}^{\infty} |\xi_i\rangle \lambda_i \langle \xi_i|; \quad \hat{\xi}_i \in L^2(\mathbf{R}^3),\ \|\hat{\xi}_i\| = (2\pi\hbar)^{-3/2} \tag{15}$$

This form is suitable for microdetectors having probabilities λ_i to be in the different internal states $\hat{\xi}_i$ before the detection is performed [14]. Probabilities that measurement outcomes of the variables $(\mathbf{q}, \mathbf{p})$ be within a domain B of Γ are then given in terms of the density matrix ρ of the physical system by

$$P_{\rho}(B) = \mathrm{Tr}[\rho \mathbf{P}_{\gamma}(B)] = \sum_{i=1}^{\infty} \lambda_i \int_{B\subset\Gamma} \langle U_{\mathbf{q},\mathbf{p}} \xi_i | \rho U_{\mathbf{q},\mathbf{p}} \xi_i \rangle d\mathbf{q} d\mathbf{p} \tag{16}$$

A modified form of expression (15) will be used in this work for test particles with intrinsic extension. Let us ignore this extension in a first step and consider extended particles whose external part is stochastically extended while the internal part is sharp.

3 Stochastically Extended Microdetectors

The most immediate implementation of stochastic values in the model of extended particles consists of applying the stochastic frame work to the external evolution of the particle. The internal evolution is considered as inaccessible to direct measurement and may be let unchanged. In other words, the stochastically extended microdetectors are used for the determination of the global location of the extended system particle in external space-time. Then the physical states belong to the Hilbert space

$$H_\xi = L^2_\xi(\mathbf{R}^6) \otimes L^2(\mathbf{R}^3) \tag{17}$$

which is a tensor product of the stochastic external and sharp internal spaces, with inner product

$$\langle \Psi | \Psi \rangle = \int_{\mathbf{R}^9} \psi^*(\mathbf{q}, \mathbf{p}; \mathbf{y}) \psi(\mathbf{q}, \mathbf{p}; \mathbf{y}) d\mathbf{q} d\mathbf{p} d\mathbf{y} \tag{18}$$

The stochastic wave function for the extended particle being

$$\psi(\mathbf{q}, \mathbf{p}; \mathbf{y}) = \langle \xi_{\mathbf{q},\mathbf{p};\mathbf{y}} | \Psi \rangle = \int_{\mathbf{R}^3} d\mathbf{x} \hat{\xi}^*_{\mathbf{q},\mathbf{p}}(\mathbf{x}) \hat{\Psi}(\mathbf{x}, \mathbf{y}) \tag{19}$$

$$|\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle = |\xi_{\mathbf{q},\mathbf{p}}\rangle \otimes |\mathbf{y}\rangle \tag{20}$$

The integral form in (19) is an isometry between H_ξ and the configuration Hilbert space $L^2(\mathbf{R}^6)$ containing the functions $\hat{\Psi}(\mathbf{x}, \mathbf{y})$. The configuration representative of $\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}$ is

$$\hat{\xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}(\mathbf{x}, \mathbf{y}') = \hat{\xi}_{\mathbf{q},\mathbf{p}}(\mathbf{x}) \delta(\mathbf{y}' - \mathbf{y}) \tag{21}$$

Rigorously, one should consider a triple $(\Phi \subset H_\xi \subset \Phi')$ consisting of a Hilbert space H_ξ, a dense subspace Φ, and the corresponding dual Φ'. We shall not go into these technical details but mention the physical interpretation of the Dirac "bra" space Φ' as that of measuring apparatuses states [4] and that $\langle \xi_{\mathbf{q},\mathbf{p};\mathbf{y}}|$ is an operator acting [16] on Φ and not on H_ξ. Since this latter remark concerns only the internal part of the present work, it is meaningful to interpret $|\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle$ as the proper state vector of a microdetector with internal pointlike degrees of freedom.

In general, a direct product of the external and internal Galilean groups G and G' is assumed to act on H_ξ through a phase space representation $U(G)$ and an induced configuration representation $\hat{U}'(G')$, respectively. A system of covariance with respect to $U(G) \otimes \hat{U}'(G')$ can be defined by

$$\mathbf{P}_\xi(B \times B') = \int_{B,B'} |\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle d\mathbf{q} d\mathbf{p} d\mathbf{y} \langle \xi_{\mathbf{q},\mathbf{p};\mathbf{y}}| \tag{22}$$

$$= \int_B |\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q} d\mathbf{p} \langle \xi_{\mathbf{q},\mathbf{p}}| \otimes \int_{B'} |\mathbf{y}\rangle d\mathbf{y} \langle \mathbf{y}| \tag{23}$$

where B is a Borel set in the external phase space Γ and B' another set in the internal configuration space $\mathbf{R}^3$. The second equality shows the product form of an external system of covariance with respect to $U(G)$ with an internal system of imprimitivity with respect to $\hat{U}'(G')$. Hence, the physical interpretations of the mean value

$$P_\Psi^\xi(B\times B') = \langle\Psi|\mathbf{P}_\xi(B\times B')\Psi\rangle = \int_{B'} d\mathbf{y}\int_B d\mathbf{q}d\mathbf{p}|\psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2 \tag{24}$$

are different for the external and internal parts in the sense that the former yields probabilities of stochastic *measurement outcomes* while the latter yields the probability that the internal mode be *effectively localized* in the region B'. Accordingly, the following marginal components

$$\mathbf{P}_\xi(B) \equiv \mathbf{P}_\xi(B\times\mathbf{R}^3) = \int_B |\xi_{\mathbf{q},\mathbf{p}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p}}|\otimes\mathbf{1} \tag{25}$$

$$\mathbf{P}_\xi(B') \equiv \mathbf{P}_\xi(\Gamma\times B') = \mathbf{1}\otimes\int_{B'}|\mathbf{y}\rangle d\mathbf{y}\langle\mathbf{y}| \tag{26}$$

are the aforementioned systems of covariance with respect to $U(G)\equiv U(G)\otimes\mathbf{1}$ and system of imprimitivity with respect to $U'(G')\equiv\mathbf{1}\otimes U'(G')$. They correspond to the respective probabilities

$$P_\Psi^\xi(B\times\mathbf{R}^3) = \langle\Psi|\mathbf{P}_\xi(B\times\mathbf{R}^3)\Psi\rangle = \int_{\mathbf{R}^3} d\mathbf{y}\int_B d\mathbf{q}d\mathbf{p}|\psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2 \tag{27}$$

$$P_\Psi^\xi(\Gamma\times B') = \langle\Psi|\mathbf{P}_\xi(\Gamma\times B')\Psi\rangle = \int_{B'} d\mathbf{y}\int_\Gamma d\mathbf{q}d\mathbf{p}|\psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2 \tag{28}$$

The operator (25) is identical to (3) and its own marginal components are those of the stochastic theory. In our case, these marginal components

$$\mathbf{P}_\xi(\Delta_1) \equiv \mathbf{P}_\xi((\Delta_1\times\mathbf{R}^3)\times\mathbf{R}^3) = \int_{\Delta_1} d\mathbf{q}\int_{\mathbf{R}^3} d\mathbf{p}|\xi_{\mathbf{q},\mathbf{p}}\rangle\langle\xi_{\mathbf{q},\mathbf{p}}|\otimes\mathbf{1} \tag{29}$$

$$\mathbf{P}_\xi(\Delta_2) \equiv \mathbf{P}_\xi((\mathbf{R}^3\times\Delta_2)\times\mathbf{R}^3) = \int_{\mathbf{R}^3} d\mathbf{q}\int_{\Delta_2} d\mathbf{p}|\xi_{\mathbf{q},\mathbf{p}}\rangle\langle\xi_{\mathbf{q},\mathbf{p}}|\otimes\mathbf{1} \tag{30}$$

give the probabilities

$$\begin{aligned}\langle\Psi|\mathbf{P}_\xi(\Delta_1)\Psi\rangle &= \int_{\Delta_1} d\mathbf{q}\int_{\mathbf{R}^6} d\mathbf{p}d\mathbf{y}|\psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2\\ &= \int_{\Delta_1} d\mathbf{q}\int_{\mathbf{R}^3} d\mathbf{x}\hat{\chi}_{\mathbf{q}}^\xi(\mathbf{x})\int_{\mathbf{R}^3} d\mathbf{y}|\hat{\Psi}(\mathbf{x},\mathbf{y})|^2\end{aligned} \tag{31}$$

$$\begin{aligned}\langle\Psi|\mathbf{P}_\xi(\Delta_2)\Psi\rangle &= \int_{\Delta_2} d\mathbf{p}\int_{\mathbf{R}^6} d\mathbf{q}d\mathbf{y}|\psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2\\ &= \int_{\Delta_2} d\mathbf{p}\int_{\mathbf{R}^3} d\mathbf{k}\tilde{\chi}_{\mathbf{p}}^\xi(\mathbf{k})\int_{\mathbf{R}^3} d\mathbf{y}|\tilde{\Psi}(\mathbf{k},\mathbf{y})|^2\end{aligned} \tag{32}$$

that the measurement of only the stochastic position or momentum of the intrinsically extended particle yield the result $\mathbf{q}\in\Delta_1$ or $\mathbf{p}\in\Delta_2$ with confidence functions

$$\hat{\chi}_{\mathbf{q}}^\xi(\mathbf{x}) = (2\pi\hbar)^3|\hat{\xi}(\mathbf{x}-\mathbf{q})|^2 \tag{33}$$

$$\tilde{\chi}^{\xi}_{\mathbf{p}}(\mathbf{k}) = (2\pi\hbar)^3|\tilde{\xi}(\mathbf{k}-\mathbf{p})|^2 \tag{34}$$

We note that the final result is the same as for the ordinary stochastic quantum mechanics except that the probability density $|\hat{\psi}(\mathbf{x})|^2$ of a pointlike system particle has been replaced by the marginal probability density $\int_{\mathbf{R}^3} d\mathbf{y}|\hat{\Psi}(\mathbf{x},\mathbf{y})|^2$ of an intrinsically extended system particle. When $\hat{\Psi}(\mathbf{x},\mathbf{y}) = \hat{\psi}(\mathbf{x})\hat{u}(\mathbf{y})$, we have $\psi(\mathbf{q},\mathbf{p};\mathbf{y}) = \psi(\mathbf{q},\mathbf{p})u(\mathbf{y})$ and the stochastic probabilities are recovered if u is normalized ($\|\hat{u}\|^2 = 1$), or by a renormalization of the confidence function which becomes

$$\hat{\chi}^{\xi}_{\mathbf{q}}(\mathbf{x}) = (2\pi\hbar)^3\|\hat{u}\|^2|\hat{\xi}(\mathbf{x}-\mathbf{q})|^2 \tag{35}$$

In studying propagation, we have to introduce the external and internal time parameters t and y^0. According to the functional theory, t designates the time measured by an observer while y^0 represents a time variable which is an argument of u with no specific physical interpretation. Denoting $y = (y^0, \mathbf{y})$, we can define the following propagator

$$K_\xi(t', \mathbf{q}', \mathbf{p}', y'; t, \mathbf{q}, \mathbf{p}, y) = \langle \xi_{\mathbf{q}',\mathbf{p}';\mathbf{y}'}|\mathcal{U}\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle \tag{36}$$

$$= \langle \xi_{\mathbf{q}',\mathbf{p}'}|U_{(t'-t)}\xi_{\mathbf{q},\mathbf{p}}\rangle\langle \mathbf{y}'|\hat{U}'_{(y'^0-y^0)}|\mathbf{y}\rangle \tag{37}$$

$$\mathcal{U} = U_{(t'-t)} \otimes \hat{U}'_{(y'^0-y^0)} \tag{38}$$

The external evolution operator is expressed in the usual way

$$U_{(t'-t)} = \exp\frac{-i}{\hbar}H_0(t'-t) \tag{39}$$

in terms of the external Hamiltonian H_0. The internal evolution operator depends on the model adopted for the physical wave u. In the case where it represents an ordinary quantum pointlike mode evolving in the internal space, the operator is

$$\hat{U}'_{(y'^0-y^0)} = \exp\frac{-i}{\hbar}\hat{H}'_0(y'^0 - y^0) \tag{40}$$

and $\hat{H}'_0$ is the internal configuration representation Hamiltonian. The total propagator is then a product of the external free stochastic propagator [14]

$$K_\xi(t', \mathbf{q}', \mathbf{p}'; t, \mathbf{q}, \mathbf{p}) = \int_{\mathbf{R}^3} d\mathbf{k}\exp\left[\frac{i\mathbf{k}^2(t-t')}{2m\hbar}\right]\tilde{\xi}^*_{\mathbf{q}',\mathbf{p}'}(\mathbf{k})\tilde{\xi}_{\mathbf{q},\mathbf{p}}(\mathbf{k}) \tag{41}$$

and the internal free pointlike propagator [12]

$$\hat{\Pi}(y'-y) = \left(\frac{\mu}{2\pi i\hbar(y'^{\circ}-y^{\circ})}\right)^{\frac{3}{2}}\exp\left\{\frac{i\mu(\mathbf{y}'-\mathbf{y})^2}{2\hbar(y'^{\circ}-y^{\circ})}\right\} \tag{42}$$

The parameters m and μ stand for the external and internal masses, respectively.

The above result is in keeping with our previous works on nonstochastically extended particles [10], but other models should not be discarded as will be done in the next section concerned with microdetectors which are stochastically and intrinsically extended.

4 Intrinsically and Stochastically Extended Microdetectors

We now endow the test particle with a distribution in the internal space so that it may be in a proper state $|\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle$ with probability density $|\hat{\lambda}(\mathbf{y})|^2$, where $\hat{\lambda}(\mathbf{y})$ is its physical wave. By analogy with (14) and (15), the system of covariance for such a particle with an internal structure is

$$\mathbf{P}_\Xi(B) = \int_{\mathbf{R}^3} d\mathbf{y}|\hat{\lambda}(\mathbf{y})|^2 \int_B |\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}| \tag{43}$$

$$\langle\hat{\xi}_\mathbf{y}|\hat{\xi}_{\mathbf{y}'}\rangle = (2\pi\hbar)^{-3}\delta(\mathbf{y}-\mathbf{y}') \tag{44}$$

It corresponds to the replacement of the discrete index in (15) by the continuous variable $\mathbf{y}$.

Now, a measurement of position and momentum, carried out with these new test particles on a system particle in a state $\rho = |\Psi\rangle\langle\Psi|$, gives values $(\mathbf{q},\mathbf{p})$ in the set $B \subset \Gamma$ with probability

$$P(B) = \langle\Psi|\mathbf{P}_\Xi(B)\Psi\rangle \tag{45}$$

$$= \int_{\mathbf{R}^3} d\mathbf{y}|\hat{\lambda}(\mathbf{y})|^2 \int_B \langle\Psi|\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle d\mathbf{q}d\mathbf{p}\langle\xi_{\mathbf{q},\mathbf{p};\mathbf{y}}|\Psi\rangle \tag{46}$$

with a confidence function to be determined later. Writing the scalar products in the configuration representation and rearranging the terms adequately, we get

$$P(B) = \int_B d\mathbf{q}d\mathbf{p}\int_{\mathbf{R}^3} d\mathbf{y}\hat{\Psi}^*(\mathbf{q},\mathbf{p};\mathbf{y})\hat{\Psi}(\mathbf{q},\mathbf{p};\mathbf{y}) \tag{47}$$

where

$$\Psi(\mathbf{q},\mathbf{p};\mathbf{y}) = \int_{\mathbf{R}^6} d\mathbf{x}d\mathbf{y}'\ \hat{\Xi}^*_{\mathbf{q},\mathbf{p};\mathbf{y}}(\mathbf{x},\mathbf{y}')\hat{\Psi}(\mathbf{x},\mathbf{y}') = \psi(\mathbf{q},\mathbf{p};\mathbf{y})\hat{\lambda}^*(\mathbf{y}) \tag{48}$$

$$\hat{\Xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}(\mathbf{x},\mathbf{y}') = \hat{\xi}_{\mathbf{q},\mathbf{p}}(\mathbf{x})\hat{\lambda}(\mathbf{y})\delta(\mathbf{y}'-\mathbf{y}) \tag{49}$$

The function $\psi(\mathbf{q},\mathbf{p};\mathbf{y})$ is given in (19). Now, Ψ can be regraded as a stochastic functional wave and $\Xi_\mathbf{y}$ as a functional proper state vector of a test particle which is intrinsically and stochastically extended. In terms of vectors, we have

$$|\Xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle = |\xi_{\mathbf{q},\mathbf{p}}\rangle \otimes \hat{\lambda}(\mathbf{y})|\mathbf{y}\rangle \tag{50}$$

$$\Psi(\mathbf{q},\mathbf{p};\mathbf{y}) = \langle\Psi|\Xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle \tag{51}$$

Then, the marginal probabilities of the stochastic theory are recovered for $\mathbf{q} \in \Delta_1$ and $\mathbf{p} \in \Delta_2$

$$\langle\Psi|\mathbf{P}_\Xi(\Delta_1)\Psi\rangle = \int_{\Delta_1} d\mathbf{q}\int_{\mathbf{R}^6} d\mathbf{x}d\mathbf{y}\hat{\chi}^\Xi_\mathbf{q}(\mathbf{x};\mathbf{y})|\hat{\Psi}(\mathbf{x},\mathbf{y})|^2 \tag{52}$$

$$\langle\Psi|\mathbf{P}_\Xi(\Delta_2)\Psi\rangle = \int_{\Delta_2} d\mathbf{p}\int_{\mathbf{R}^6} d\mathbf{k}d\mathbf{y}\tilde{\chi}^\Xi_\mathbf{p}(\mathbf{k};\mathbf{y})|\tilde{\Psi}(\mathbf{k},\mathbf{y})|^2 \tag{53}$$

The new confidence functions

$$\hat{\chi}^\Xi_\mathbf{q}(\mathbf{x};\mathbf{y}) = (2\pi\hbar)^3|\hat{\Xi}_\mathbf{y}(\mathbf{x}-\mathbf{q})|^2 = (2\pi\hbar)^3|\hat{\xi}(\mathbf{x}-\mathbf{q})|^2|\hat{\lambda}(\mathbf{y})|^2 \tag{54}$$

$$\tilde{\chi}^\Xi_\mathbf{p}(\mathbf{k};\mathbf{y}) = (2\pi\hbar)^3|\tilde{\Xi}_\mathbf{y}(\mathbf{k}-\mathbf{p})|^2 = (2\pi\hbar)^3|\tilde{\xi}(\mathbf{k}-\mathbf{p})|^2|\hat{\lambda}(\mathbf{y})|^2 \tag{55}$$

are conditional probabilities that the measurement of the stochastic position and momentum yield the results $\mathbf{q}$ and $\mathbf{p}$ when the external mode of the particle is at position $\mathbf{x}$ with momentum $\mathbf{k}$ and the internal mode is located at $\mathbf{y}$. Due to the product forms of $\hat{\Xi}$ and $\hat{\chi}$, we reach the same conclusion in this case as that in the preceding section whenever the functional $\hat{\Psi}(\mathbf{x},\mathbf{y}) = \hat{\psi}(\mathbf{x})u(\mathbf{y})$, namely, probabilities are identical to the stochastic ones with suitable renormalizations.

All the above results sustain the interpretation of $\Xi_{\mathbf{y}}$ as a proper state vector of a test particle, at stochastic rest at the origin of a classical inertial frame, which is in the internal state labeled by $\mathbf{y}$. Consequently, a stochastic theory can be reconstructed with these new proper state vectors and the corresponding wave functions $\Psi(\mathbf{q},\mathbf{p},\mathbf{y})$. For instance, we can define the following propagator

$$K_\xi(t',\mathbf{q}',\mathbf{p}',y';t,\mathbf{q},\mathbf{p},y) = \langle \hat{\Xi}_{\mathbf{q}',\mathbf{p}';\mathbf{y}'} | \mathcal{U} \hat{\Xi}_{\mathbf{q},\mathbf{p};\mathbf{y}} \rangle \tag{56}$$

$$= \hat{\lambda}^*(\mathbf{y}')\hat{\lambda}(\mathbf{y}) \langle \hat{\xi}_{\mathbf{q}',\mathbf{p}';\mathbf{y}'} | \mathcal{U} \hat{\xi}_{\mathbf{q},\mathbf{p};\mathbf{y}} \rangle \tag{57}$$

$$\mathcal{U} = U_{(t'-t)} \otimes \hat{U}'_{(y'^0 - y^0)} \tag{58}$$

for the stochastic wave functional $\Psi_{t,y^0}(\mathbf{q},\mathbf{p};\mathbf{y}) = (U_t \otimes \hat{U}'_{y^0})\Psi(\mathbf{q},\mathbf{p};\mathbf{y})$. Respectively, the evolution operators U and $\hat{U}'$are the same as (39) and (40).

Let us note at the end of this section that analogous formulas could be derived if we started with proper state vectors $\hat{\xi}_{\mathbf{y}}$ depending on one rather than two variables. This would entail the replacement of $\hat{\xi}_{\mathbf{y}}(\mathbf{x},\mathbf{y}')$ and $\hat{\Psi}(\mathbf{x},\mathbf{y}')$ by $\hat{\xi}_{\mathbf{y}}(\mathbf{x})$ and $\hat{\psi}(\mathbf{x})$ and would correspond to the unusual situation where pointlike system particles are observed with stochastically and intrinsically extended particles. For example, relation (19) would read

$$\psi(\mathbf{q},\mathbf{p};\mathbf{y}) = \int_{\mathbf{R}^6} d\mathbf{x}\, \hat{\xi}^*_{\mathbf{q},\mathbf{p};\mathbf{y}}(\mathbf{x})\hat{\psi}(\mathbf{x}) \tag{59}$$

From an opposite standpoint, we may get rid of the δ function by choosing state vectors $\hat{\xi}_{\mathbf{q},\mathbf{p};\mathbf{y}}(\mathbf{x},\mathbf{y}') = \hat{\xi}_{\mathbf{q},\mathbf{p}}(\mathbf{x})d(\mathbf{y},\mathbf{y}')$ with bona fide d functions expressing correlations in the internal space. Then, the marginal probability for the measurement of the stochastic position $\mathbf{q}$ is

$$\langle \Psi | \mathbf{P}_\Xi(\Delta_1)\Psi\rangle = \int_{\Delta_1} d\mathbf{q} \int_{\mathbf{R}^6} d\mathbf{p} d\mathbf{y} |\Psi(\mathbf{q},\mathbf{p};\mathbf{y})|^2$$

$$= \int_{\Delta_1} d\mathbf{q} \int_{\mathbf{R}^9} d\mathbf{x} d\mathbf{y}' d\mathbf{y}'' \hat{\chi}^\Xi_{\mathbf{q}}(\mathbf{x};\mathbf{y}',\mathbf{y}'')\hat{\Psi}^*(\mathbf{x},\mathbf{y}'')\hat{\Psi}(\mathbf{x},\mathbf{y}') \tag{60}$$

$$\hat{\chi}^\Xi_{\mathbf{q}}(\mathbf{x};\mathbf{y}',\mathbf{y}'') = (2\pi\hbar)^3 \int_{\mathbf{R}^3} d\mathbf{y}\, \hat{\Xi}^*_{\mathbf{y}}(\mathbf{x}-\mathbf{q},\mathbf{y}')\hat{\Xi}_{\mathbf{y}}(\mathbf{x}-\mathbf{q},\mathbf{y}'')$$

$$= (2\pi\hbar)^3 |\hat{\xi}(\mathbf{x}-\mathbf{q})|^2 \int_{\mathbf{R}^3} d\mathbf{y} |\hat{\lambda}(\mathbf{y})|^2 d(\mathbf{y},\mathbf{y}')d(\mathbf{y},\mathbf{y}'') \tag{61}$$

This latter formulation has the advantage of the Hibert space structure simplicity avoiding the triple structure $(\Phi \subset H_\xi \subset \Phi')$ mentioned in the beginning of the previous section. However, the physical interpretation of the internal correlation function d is not quite clear at the moment. Moreover the propagator turns out to be difficult to define.

5 Conclusion

The analysis of the functional and stochastic theories revealed that the former conceptualized a particle as intrinsically extended while the latter viewed that extension from an operational standpoint as stochastic. The complementarity of these two standpoints is a natural question which has not been addressed so far. In this work, we dealt with a complete description of the extension of the particle as being intrinsically and stochastically extended. The focus was on the spinless non relativistic case for its mathematical simplicity and its clear-cut physical interpretations. We have presented two cases where the functional of the system particle is represented by a bilocal wave $\hat{\Psi}(\mathbf{x},\mathbf{y})$.

In the first case, the test particle is identical to that of the stochastic theory with no intrinsic extension. This results in a stochastic description of the external part of the bilocal wave while the internal part retains its sharp character. This is in keeping with the idea that only the external part can undergo direct experimental observation. The quantization in the internal space-time can then be performed with the method of induced representation and the corresponding systems of imprimitivity determine probabilities of actual presence in a region (i.e. not probabilities of measurement outcomes). Hilbert spaces, systems of covariance, and propagators are products of the stochastic external part and the sharp internal part. However, the wave function $\Psi(\mathbf{q},\mathbf{p},\mathbf{y})$ is a product form only when $\hat{\Psi}(\mathbf{x},\mathbf{y})$ is, with the consequence that external marginal probabilities are those of the stochastic theory modulo a normalization.

The second case is an improvement of the former where the stochastic test particle is endowed with an intrinsic extension. This implementation is not trivial since it corresponds to associating its physical wave λ to a model drawn from the (statistical) stochastic theory by interpreting $|\hat{\lambda}(\mathbf{y})|^2$ as the probability for the stochastic particle to be in the internal state $\xi_{\mathbf{y}}$. Relation (48) yields a relation between the stochastic functional wave $\Psi(\mathbf{q},\mathbf{p};\mathbf{y})$ and the configuration functional wave $\hat{\Psi}(\mathbf{x},\mathbf{y})$ and shows that the proper state vectors ξ can be replaced by their functional counterparts Ξ. These are given by $|\Xi_{\mathbf{q},\mathbf{p};\mathbf{y}}\rangle = |\xi_{\mathbf{q},\mathbf{p}}\rangle \otimes \hat{\lambda}(\mathbf{y})|\mathbf{y}\rangle$ and their explicit form is mandatory of that of $\hat{\lambda}$ (note that an optimal expression for $\xi_{\mathbf{q},\mathbf{p}}$ has already been proposed and a variety of ways for the determination of alternative expressions have been pointed out [14, 15]). Moreover, the propagator has a product form and the stochastic interpretations can be transferred to the functional $\Psi(\mathbf{q},\mathbf{p};\mathbf{y})$. It is the probability amplitude that the simultaneous measurement of position and momentum yields the stochastic values $(\mathbf{q},\hat{\chi}^{\Xi}_{\mathbf{q}})$ and $(\mathbf{p},\tilde{\chi}^{\Xi}_{\mathbf{p}})$ when the internal mode is at $\mathbf{y}$. This probability is not essentially different from the stochastic one when $\hat{\Psi}(\mathbf{x},\mathbf{y}) = \hat{\psi}(\mathbf{x})\hat{u}(\mathbf{y})$ and differs by a normalization constant as in the first case.

The advantage of the above construction appears in the relativistic and general relativistic regimes where the stochastic extension cures many of their fundamental inconsistencies [15] and [16]. In fact, our conception of the intrinsic extension enables the consideration of confined quantum modes which cannot be observed without the extended particle and may be applicable to hadrons composed of quarks which may play the role of internal quantum modes. In addition, the internal symmetry can be a de Sitter one which has been used in describing a relativistic rotator with an acceptable mass formula for hadrons [1]. Gauging this symmetry in a nonlinear representation is capable of describing the space-time extension of hadrons as regions where the full de Sitter connection takes place and the exterior of hadrons as the regions where the de Sitter symmetry is broken so that the Lorentz subsymmetry is linked with classical gravity [8]. This de Sitter gauge symmetry has been quantized from a purely stochastic standpoint [9] and from a purely intrinsic standpoint [11]. Our future plan is to improve the latter geometro-differential intrinsic quantum framework by incorporating the stochastic component in it through the replacement of the stochastic quantum frames corresponding to the proper state vectors ξ by new quantum frames corresponding to our proper state vectors ξ_y and Ξ. This work is under consideration and may lead to a consistent theory of hadrons in a classical gravitational background. Quantum gravity, which constitutes our final goal, will thereafter be considered along the lines of the geometro-stochastic theory. We hope that other researchers much more acquainted with quark models will find concrete applications of the present work.

References

1. Aldinger, R.R., et al.: Relativistic rotator. I. Quantum observables and constrained Hamiltonian mechanics. Phys. Rev. D **28**, 3020–3031 (1983)
2. Ali, S.T., Prugovečki, E.: Systems of imprimitivity and representations of quantum mechanics on fuzzy phase space. J. Math. Phys. **18**, 219–228 (1977)
3. Ali, S.T., Prugovečki, E.: Extended harmonic analysis of phase space representations for the Galilean group. Acta Appl. Math. **6**, 19–45 (1986)
4. Antoine, J.-P.: Dirac formalism and symmetry problems in quantum mechanics. I. General Dirac formalism. J. Math. Phys. **10**, 53–69 (1969)
5. de Broglie, L.: La Réinterprétation de la Mécanique Ondulatoire. Gauthier-Villars, Paris (1971)
6. Destouches, J.L.: La Quantification en Théorie Fonctionnelle des Corpuscules. Gauthier-Villars, Paris (1956)
7. Destouches, J.L.: Corpuscules et Champs en Théorie Fonctionnelle. Gauthier-Villars, Paris (1958)
8. Drechsler, W.: Modified Weyl theory and extended elementary objects. Found. Phys. **19**, 1479–1497 (1989)
9. Drechsler, W.: Quantized fiber dynamics for extended elementary objects involving gravitation. Found. Phys. **22**, 1041–1077 (1992)
10. Hachemane, M., Benbitour, M., Smida, A.: Geometro-differential conception of extended particles and the Galilean semigroup of trajectories. Found. Phys. **27**, 579–594 (1997)
11. Hachemane, M., Smida, A., Djelid, R.: Functional realization and nonlinear induced representation in the geometro-differential conception of extended particles. Found. Phys. **29**, 1479–1494 (1999)
12. Mensky, M.B.: The Method of Induced Representations: Space-Time and the Concept of Particles. Nauka, Moscow (1976) (in Russian)
13. Prugovečki, E.: Probability measures on fuzzy events in phase space. J. Math. Phys. **17**, 517–523 (1977)
14. Prugovečki, E.: Stochastic Quantum Mechanics and Quantum Space-Time. Reidel, Dordrecht (1984)
15. Prugovečki, E.: Quantum Geometry. Kluwer Academic, Dordrecht (1992)
16. Prugovečki, E.: Principles of Quantum General Relativity. World Scientific, Singapore (1995)
17. Smida, A., Hachemane, M., Fellah, M.: Geometro-differential conception of extended particles and their quantum theory in de Sitter space. Found. Phys. **25**, 1769–1795 (1995)